Expansions for Droplet States in the Ferromagnetic XXZ Heisenberg Chain

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Abstract

We consider the highly anisotropic ferromagnetic spin 1/2 Heisenberg chain with periodic boundary conditions. In each sector of constant total z component of the spin, we develop convergent expansions for the lowest band of eigenvalues and eigenfunctions. These eigenstates describe droplet states in which the spins essentially form a single linear droplet which can move. Our results also give a convergent expansion for the dispersion relation, i.e., the energy of the droplet as a function of its momentum. The methods used are from [4] and [5], and this short paper should serve as a pedagogic introduction to those papers.

1 Introduction

We consider the spin 1/2 ferromagnetic XXZ Heisenberg chain. With periodic boundary conditions its Hamiltonian is

$$H = -\sum_{j=1}^{N} [\sigma_j^z \sigma_{j+1}^z + \epsilon \sigma_j^x \sigma_{j+1}^x + \epsilon \sigma_j^y \sigma_{j+1}^y] \tag{1}$$

We will always assume that ϵ is small, so we are in the Ising-like regime. This Hamiltonian commutes with the total z component of the spin, so one can study it in a sector with a fixed total z component of spin. The lowest energy states in each such sector are droplet states in which the down spins essentially form a single droplet. This droplet can move around, so the lowest part of the spectrum in each sector should be a band of continuous spectrum. Much of this picture was proved by Nachtergaele and Starr [14]. In particular, they showed that the eigenvalues of these droplet states lie in an interval whose width is of order ϵ^m where m is the number of spins in the droplet, and there is a gap between this band and the rest of the spectrum. They gave an explicit description of the subspace spanned by these droplet eigenstates which becomes exact in the limit that the size of the droplet goes to infinity.

We will show that methods from [4] and [5] can be used to construct convergent expansions for the eigenstates in this lowest band in each sector with a fixed number of down spins. Because the translation-invariant ground states of the ferromagnetic XXZ chain are trivial, the estimates in this paper are considerably simpler than those in [4] and [5]. Thus this short paper should serve as a pedagogic introduction to those papers. At first glance it may be surprising that one can construct convergent expansions for eigenstates that will become part of continuous spectrum in the limit that the length of the chain goes to infinity. We avoid this problem by using the fact that with periodic boundary conditions the eigenfunctions have a definite momentum. Within the subspace of a given momentum the eigenvalue we want to study is isolated, and a convergent expansion is possible. The results of [14] are valid for $|\epsilon| < 1$, while our results are valid only for small ϵ. So our results have a smaller range of applicability, but give more detail about the droplet eigenstates and eigenvalues.

Before we consider the droplet states, we will first consider the chain with the periodic boundary conditions replaced by boundary conditions that favor the spin to be up at one end of the chain and down at the other end of the chain. Such boundary conditions force a kink or interface into the chain. For a generic Hamiltonian one would not expect this interface to be stable. Roughly speaking it would be equally likely to be anywhere along the chain. However, for the model considered here the conservation of total z component of spin essentially fixes the location of the kink. The stability of this interface was proved independently by Alcaraz, Salinas and Wreszinski [1] and Gottstein and Werner [6]. Alternative proofs of the stability of this interface were given in [3], by using the path integral representation of interface states, and in [2]. Other work on these interface states includes [7], [8], [9], [10], [11], and [12]. Our discussion of these interface states does not contain any new results and is intended as an introduction to the methods used for developing the expansion of the droplet states.

We will always work on finite chains with estimates independent of the length of the chain. Of course, one obtains continuous spectra only in the infinite length limit. It would be interesting to show that in the infinite length limit, defined for example via the GNS construction, there is indeed a band of continuous spectrum corresponding to the motion of the droplets.

This paper only concerns the one-dimensional XXZ ferromagnet. There are many interesting results on interfaces in the higher dimensional model (see [13] for a review), but we are not aware of any results on droplet states in the higher dimensional model. In two dimensions, if m, the number of spins in the droplet, is a perfect square, then the ground states of the $\epsilon = 0$ Hamiltonian are just droplet states in which the droplet is a square. So the $\epsilon = 0$ ground state is unique up to translations. In these sectors it may be possible to use the methods of this paper to contruct the droplet eigenstates. (A similar remark applies in higher dimensions.) For other sectors the $\epsilon = 0$ ground states are typically more degenerate, and the methods of this paper would need to be combined with some form of degenerate perturbation theory.

2 Kink states

In this section we construct convergent expansions for the "kink states" which contain a single domain wall. These states have been studied extensively, and there are no new results in this section. This section serves as an introduction to our expansions for the droplet states in the next section. The expansion for the kink states is similar to that for the droplet states, but simpler since one does not have to deal with the freedom of the droplets to move about.

We consider the ferromagnetic XXZ Heisenberg chain on N sites with the following boundary conditions.

$$H = \sum_{j=1}^{N-1} [1 - \sigma_j^z \sigma_{j+1}^z - \epsilon \sigma_j^x \sigma_{j+1}^x - \epsilon \sigma_j^y \sigma_{j+1}^y] + A(1 + \sigma_1^z) + B(1 - \sigma_N^z) \tag{2}$$

The constants A and B will be positive, so the boundary conditions favor the spin down state at site 1 and the spin up state at site N. If we consider the sector with m down spins, then the above Hamiltonian with $\epsilon = 0$ has a unique ground state given by putting the m down spins at sites 1 to m and the up spins at the other sites. For a set X of sites we define $|X\rangle$ to be the state with down spins at the sites in X and up spins everywhere else. So the $\epsilon = 0$ ground state is $|\{1, 2, \cdots, m\}\rangle$. The set $\{1, 2, \cdots, m\}$ will appear throughout our equations, so we denote it by

$$M = \{1, 2, \cdots, m\}. \tag{3}$$

When ϵ is not zero, the ground state in the sector with m down spins will be a perturbation of $|M\rangle$. The coefficient of $|Y\rangle$ should get smaller as Y gets "farther" from M. Thus it is natural to write Y as a modification of M. For sets of sites X and Y we denote by $X \Delta Y$ the set of sites which belong to exactly one of X and Y. So

$$X \Delta Y = (X \cup Y) \setminus (X \cap Y) \tag{4}$$

Any state in the sector with m down spins can be written as

$$\Psi = \sum_{X:m} e(X) \, |X\Delta M\rangle \tag{5}$$

for some coefficients $e(X)$. The constraint $X : m$ means that we only sum over sets X such that $|X\Delta M\rangle$ has m down spins. This means that $X\Delta M$ has m elements, or equivalently that $X \cap M$ and $X \cap M^c$ have the same number of elements. For example, X could be $\{m, m+1\}$ in which case $|M\Delta X\rangle$ is the state that occurs at first order in perturbation theory.

When $\epsilon = 0$, $|X\Delta M\rangle$ is an eigenstate for all X. We let $\lambda(X)$ denote its eigenvalue. It equals 2 times the number of nearest neighbor pairs of sites for which the spins in $|X\Delta M\rangle$ are anti-parallel plus boundary terms. The boundary terms are $+2A$ if $1 \in X$ and $+2B$ if $N \in X$. For the groundstate, $X = \emptyset$, we have $\lambda(X) = 2$. If we have $A \geq 1$ and $B \geq 1$, then for all other X with $X : m$ we have $\lambda(X) \geq 6$. (Note that there are other X with $\lambda(X) = 2$, e.g., $X = \{m+1\}$, but they belong to sectors with a different number of down spins.) If A and B are positive, but possibly less than 1, then the lower bound of 6 must be replaced by the minimum of 6 and $2 + 2A + 2B$. The following proof applies in this case with slight modifications. To keep things as simple as possible we will assume A and B are both at least 1. (In the more general case, just how small ϵ would need to be would depend on A and B.)

We now consider the Hamiltonian with nonzero ϵ. The term $\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y$ can be written as $\sigma_j^x \sigma_{j+1}^x (1 - \sigma_j^z \sigma_{j+1}^z)$. So if the spins at j and $j+1$ are parallel, it gives zero and if they are anti-parallel it flips these two spins and multiplies the result by 2. We let $\partial(X\Delta M)$ denote the set of bonds $< j, j+1 >$ such that the spins at j and $j+1$ in $|X\Delta M\rangle$ are different. One can think of $\partial(X\Delta M)$ as the set of domain walls in the spin configuration. Then

$$H\Psi = \sum_{X:m} \lambda(X) e(X) \, |X\Delta M\rangle + 2\epsilon \sum_{X:m} e(X) \sum_{<j,j+1> \in \partial(X\Delta M)} |\{j, j+1\}\Delta X\Delta M\rangle \tag{6}$$

Note that $< j, j+1 > \in \partial(X\Delta M)$ if and only if $< j, j+1 > \in \partial(\{j, j+1\}\Delta X\Delta M)$. So we can do a change of variables $X \to X\Delta\{j, j+1\}$ in the last term and rewrite this as

$$H\Psi = \sum_{X:m} \lambda(X) e(X) \, |X\Delta M\rangle + 2\epsilon \sum_{X:m} \sum_{<j,j+1> \in \partial(X\Delta M)} e(X\Delta\{j, j+1\}) \, |X\Delta M\rangle \tag{7}$$

When $\epsilon = 0$, the ground state energy is 2, so we will write the ground state energy for nonzero ϵ in the form $2 + E$, i.e, we look for a solution of $H\Psi = (2 + E)\Psi$. Equating the coefficients of $|X\Delta M\rangle$, we have for X with $X : m$,

$$\lambda(X) e(X) + 2\epsilon \sum_{<j,j+1> \in \partial(X\Delta M)} e(X\Delta\{j, j+1\}) = (2 + E) e(X) \tag{8}$$

We take $e(\emptyset) = 1$. (At first we cannot be sure this is possible since $e(\emptyset)$ could be zero for the groundstate. But if we can succeed in constructing an eigenstate with this condition and

show it is the ground state, then this will justify the assumption.) The unknowns in the above equation are $e(X)$ for non-empty X with $X : m$ and E. For $X = \emptyset$ the above equation gives

$$E = 2\epsilon\, e(\{m, m+1\}) \tag{9}$$

and for non-empty X with $X : m$ the above equation can be written as

$$e(X) = \frac{1}{\lambda(X) - 2} \left[-2\epsilon \sum_{<j,j+1> \in \partial(X \Delta M)} e(X \Delta \{j, j+1\}) + Ee(X) \right] \tag{10}$$

We think of equations (9) and (10) as a fixed point equation for the unknowns E and $e(X)$, where X ranges over non-empty sets with $X : m$. We use e to denote this collection of unknowns and write equations (9) and (10) together as $F(e) = e$. We define a norm on the space of e's by

$$||e|| = |E| + \sum_{X:m, X \neq \emptyset} (\lambda(X) - 2)|e(X)| \tag{11}$$

We will show that F is a contraction in a small ball about the origin and maps this ball back into itself. The existence of a solution then follows by the contraction mapping theorem.

At first glance it seems that $F(0) = 0$, but it is not. This is because $e(\emptyset)$ can appear in the right side of (10). This happens when $X = \{m, m+1\}$ and $j = m$. This produces a "constant" term in $F(e)$, i.e., a term which does not depend on e. Denoting the terms in $F(0)$ by $e_0(X)$ we see that $e_0(\{m, m+1\}) = -\epsilon/2$, and the other $e_0(X)$ and E_0 are all zero. We note, for later use, that $||F(0)|| = 2\epsilon$.

To show that F is a contraction we have

$$\begin{aligned} ||F(e) - F(e')|| &\leq 2\epsilon |e(\{m, m+1\}) - e'(\{m, m+1\})| \\ &+ 2\epsilon \sum_{X:m, X \neq \emptyset} \sum_{<j,j+1> \in \partial(X \Delta M)} |e(X \Delta \{j, j+1\}) - e'(X \Delta \{j, j+1\})| \\ &+ \sum_{X:m, X \neq \emptyset} |Ee(X) - E'e'(X)| \end{aligned} \tag{12}$$

We split this bound into two parts. The first term in the right side of the above is just what the second term would give when $X = \emptyset$. So the first two terms of the three terms in the right side of the above are

$$\begin{aligned} &= 2\epsilon \sum_{X:m} \sum_{<j,j+1> \in \partial(X \Delta M)} |e(X \Delta \{j, j+1\}) - e'(X \Delta \{j, j+1\})| \\ &= 2\epsilon \sum_{X:m} \lambda(X) |e(X) - e'(X)| \end{aligned} \tag{13}$$

where the last equality follows by a change of variables, $X \to X \Delta \{j, j+1\}$. For $X = \emptyset$, $|e(X) - e'(X)| = 0$. For all other X with $X : m$ we have $\lambda(X) \geq 6$ and so $\lambda(X) \leq \frac{3}{2}(\lambda(X) - 2)$. Thus the above is

$$\leq 3\epsilon ||e - e'|| \tag{14}$$

The second part of (12) is

$$\begin{aligned}\sum_{X:m,X\neq\emptyset} |Ee(X) - E'e'(X)| &\leq \sum_{X:m,X\neq\emptyset} [\,|E|\;|e(X)-e'(X)| + |E-E'|\;|e'(X)|\,] \\ &\leq \frac{1}{4}\sum_{X:m,X\neq\emptyset} (\lambda(X)-2)\,[\,|E|\;|e(X)-e'(X)| + |E-E'|\;|e'(X)|\,] \\ &\leq \frac{1}{4}||e-e'||\quad \max\{||e||,||e'||\} \end{aligned} \tag{15}$$

where we have used $\lambda(X) - 2 \geq 4$.

Putting together these two bounds, we have shown for e,e' with norm less than δ that

$$||F(e) - F(e')|| \leq (3\epsilon + \frac{1}{4}\max\{||e||,||e'||\})||e-e'|| \leq (3\epsilon + \frac{\delta}{4})||e-e'|| \tag{16}$$

Thus F is a contraction on the ball of radius δ about the origin if δ and ϵ are small enough. To see that it maps the ball of radius δ about the origin back into itself, we use

$$||F(e)|| \leq ||F(0)|| + ||F(e) - F(0)|| \leq 2\epsilon + (3\epsilon + \frac{\delta}{4})||e-0|| < \delta \tag{17}$$

if $||e|| < \delta$ and ϵ is sufficiently small.

By using a stronger norm we can derive decay properties for the coefficients $e(X)$. Let K be a large positive constant. Define

$$||e|| = |E|(K|\epsilon|)^{-2} + \sum_{X:m,X\neq\emptyset} (\lambda(X)-2)|e(X)|(K|\epsilon|)^{-w(X)} \tag{18}$$

We will assume that ϵ is small enough that $K|\epsilon| < 1$. $w(X)$ is a positive integer which is the lowest order in perturbation theory at which $e(X)$ gets a nonzero contribution. More precisely, we consider all sequences of sets $X_0, X_1, \cdots, X_n$ such that $X_0 = \emptyset$, $X_n = X$ and for $i = 1, 2, \cdots, n$, $X_i = X_{i-1}\Delta\{j, j+1\}$ for some $j \in \partial(X_{i-1}\Delta M)$. Then $w(X)$ is the smallest n for which such a sequence exists.

The previous estimates that proved the existence of a fixed point in our original norm can be repeated with this new norm. One finds that the estimates continue to hold provided $w(X)$ satisfies the following three properties.

$$\begin{aligned} &w(\{m, m+1\}) = 1 \\ &w(X\Delta\{j, j+1\}) \leq w(X) + 1 \\ &w(X) - 2 \leq w(X) \end{aligned} \tag{19}$$

The first two properties follow easily from the definition of $w(X)$, while the third is trivial. Note that the existence of a solution to the fixed point equation in this stronger norm implies that the coefficients $e(X)$ decay at least as fast as $(K|\epsilon|)^{w(X)}$.

The expansion that we have developed can now be used to study a variety of properties of the kink states. For example, the localization of the kink near the site m follows from these estimates. One can consider the dependence of the ground state energy on the sector (the choice of m). Most studies of the kink states used particular values of A and B for which this energy is independent of the sector. Bach and Macris [2] considered more general boundary conditions and showed that the difference between the ground state energies in different sectors were exponentially small in the length of the chain provided the kink is not near the boundaries. With more work it is probably possible to rederive this result with our expansion. Our interest here is primarily in the droplet states, so we do not pursue this approach to the kink states any further.

3 Droplet states

Now we consider the ferromagnetic XXZ Heisenberg chain with periodic boundary conditions. For N sites its Hamiltonian may be taken to be

$$H = \sum_{j=1}^{N} [1 - \sigma_j^z \sigma_{j+1}^z - \epsilon \sigma_j^x \sigma_{j+1}^x - \epsilon \sigma_j^y \sigma_{j+1}^y] \tag{20}$$

Indices will always be taken to be periodic, e.g., σ_{N+1}^z means σ_1^z.

We consider the sector with m down spins. We continue to use the abbreviation

$$M = \{1, 2, \cdots, m\} \tag{21}$$

For $\epsilon = 0$ the ground states in this sector are $|M\rangle$ and its translates. When $\epsilon \neq 0$, the interface between sites m and $m+1$ and between sites 1 and N will spread out somewhat. States of the form $|M \Delta X\rangle$ where X is small and localized near 1 and m will make up the dominant part of the eigenstate. We continue to denote the constraint that $|X \Delta M\rangle$ is in the sector with m down spins by $X : m$. We now look for eigenstates with momentum k in the form

$$\Psi_k = \sum_{l=1}^{N} e^{ikl} \sum_{X:m} e(X) \ |(X \Delta M) + l\rangle \tag{22}$$

For a set of sites Y, we use $Y + l$ to denote the translate of the set by l sites to the right. So $Y + l = \{i + l : i \in Y\}$. This state has momentum k in the sense that if T is the operator of translation by one lattice site to the right, then $T\Psi_k = e^{-ik}\Psi_k$.

As before, ∂X is the set of bonds such that one endpoint is in X and the other endpoint is not in X. We define $n(X)$ to be the number of bonds in $\partial(X \Delta M)$. Then $2n(X)$ is the eigenvalue of $|X \Delta M\rangle$ when $\epsilon = 0$. We have

$$\begin{aligned} H\Psi_k &= 2 \sum_{l=1}^{N} e^{ikl} \sum_{X:m} e(X) n(X) \ |(X \Delta M) + l\rangle \\ &\quad + 2\epsilon \sum_{l=1}^{N} e^{ikl} \sum_{X:m} e(X) \sum_{<j,j+1> \in \partial((X \Delta M)+l)} |\{j, j+1\} \Delta [(X \Delta M) + l]\rangle \end{aligned} \tag{23}$$

Using the change of variables $j \to j+l$ in the second term, this is

$$\begin{aligned} &= 2\sum_{l=1}^{N} e^{ikl} \sum_{X:m} e(X)n(X)\ |(X\Delta M)+l\rangle \\ &\quad +2\epsilon \sum_{l=1}^{N} e^{ikl} \sum_{X:m} \sum_{<j,j+1>\in\partial(X\Delta M)} e(X)\ |(\{j,j+1\}\Delta X\Delta M)+l\rangle \end{aligned} \tag{24}$$

In the second sum on X we do a change of variables: $X \to X\Delta\{j,j+1\}$. Since $< j,j+1 > \in \partial(X\Delta M)$ if and only if $< j,j+1 > \in \partial(\{j,j+1\}\Delta X\Delta M)$, we obtain

$$\begin{aligned} H\Psi_k &= 2\sum_{l=1}^{N} e^{ikl} \sum_{X:m} e(X)n(X)\ |(X\Delta M)+l\rangle \\ &\quad +2\epsilon \sum_{l=1}^{N} e^{ikl} \sum_{X:m} \sum_{<j,j+1>\in\partial(X\Delta M)} e(X\Delta\{j,j+1\})\ |(X\Delta M)+l\rangle \end{aligned} \tag{25}$$

The above should be equal to $E(k)\Psi_k$. The eigenvalue now depends on k. We write $E(k)$ as a Fourier series. Since it equals 2 when $\epsilon = 0$, we take the series in the form

$$E(k) = 2 + \sum_{s=1}^{N} e_s e^{iks} \tag{26}$$

So

$$\begin{aligned} (E(k)-2)\Psi_k &= \sum_{s=1}^{N} e_s e^{iks} \sum_{l=1}^{N} e^{ikl} \sum_{X:m} e(X)\ |(X\Delta M)+l\rangle \\ &= \sum_{s=1}^{N}\sum_{l=1}^{N} e_s e^{ikl} \sum_{X:m} e(X)\ |(X\Delta M)+l-s\rangle \end{aligned} \tag{27}$$

where we have used the change of variables $l \to l-s$. Define Y by

$$(X\Delta M)+l-s = (Y\Delta M)+l \tag{28}$$

Then solving for X in terms of Y we find

$$X = (Y+s)\Delta(M+s)\Delta M \tag{29}$$

So

$$(E(k)-2)\Psi_k = \sum_{s=1}^{N}\sum_{l=1}^{N} e_s e^{ikl} \sum_{X:m} e((Y+s)\Delta(M+s)\Delta M)\ |(Y\Delta M)+l\rangle \tag{30}$$

We now multiply both of (25) and (30) by e^{-ikn} and sum on k. This yields

$$\begin{aligned} & 2\sum_{X:m} e(X)(n(X)-2)\ |(X\Delta M)+n\rangle \\ & +2\epsilon\sum_{X:m}\sum_{j\in\partial(X\Delta M)} e(X\Delta\{j,j+1\})\ |(X\Delta M)+n\rangle \\ = & \sum_{s=1}^{N} e_s \sum_{X:m} e((Y+s)\Delta(M+s)\Delta M)\ |(Y\Delta M)+n\rangle \end{aligned} \tag{31}$$

Thus for sets X such that $X:m$ we have

$$\begin{aligned} & 2(n(X)-2)e(X)+2\epsilon\sum_{j\in\partial(X\Delta M)} e(X\Delta\{j,j+1\}) \\ & =\sum_{s=1}^{N} e_s\, e((X+s)\Delta(M+s)\Delta M) \end{aligned} \tag{32}$$

The smallest $n(X)$ can be is 2. It attains this value for the empty set and for X of the form $M\Delta(M+n)$ for some n. (These are the sets for which $X\Delta M$ is just a translate of M.) We take $e(\emptyset)=1$ and $e(X)=0$ for all other X with $n(X)=2$. It is not clear at first that we can do this, but if we can succeed in constructing the eigenfunctions under this condition that will show that these conditions can be imposed. We note that since we now have N eigenfunctions, there are N degrees of freedom corresponding to their normalizations. The conditions we impose can be thought of as fixing these normalizations.

If $X=M\Delta(M-n)$, then

$$\sum_{s=1}^{N} e_s\, e((X+s)\Delta(M+s)\Delta M)=e_n \tag{33}$$

So for these X, eq. (32) becomes

$$e_n=2\epsilon\sum_{j\in\partial(M-n)} e(M\Delta(M-n)\Delta\{j,j+1\}) \tag{34}$$

For X with $n(X)>2$, we solve eq. (32) for $e(X)$:

$$\begin{aligned} e(X) & =-\frac{\epsilon}{n(X)-2}\sum_{j\in\partial(X\Delta M)} e(X\Delta\{j,j+1\}) \\ & +\frac{1}{2}\frac{1}{n(X)-2}\sum_{s=1}^{N} e_s\, e((X+s)\Delta(M+s)\Delta M) \end{aligned} \tag{35}$$

We use e to denote the collection of variables $e(X)$ for X such that $X:m$ and $n(X)>2$ and the variables e_n. These are the unknowns in (34) and (35). The right sides of (34) and (35) define a function $F(e)$, and together these two equations can be written as the fixed point

equation $F(e) = e$. It is important to ask if the argument of any of the $e(\quad)$ in the right sides of these equations can be the empty set or a set of the form $M\Delta(M+s)$. It is easy to check this does not happen in (34), except for the trivial case of $m = 1$. And it does not happen in the second term in (35) thanks to the constraint $n(X) > 2$. But it can happen in the first term in (35). Since $e(M\Delta(M+s)) = 0$ for $s \neq 0$, these terms drop out of (35). And since $e(\emptyset) = 1$, when $X = \{m, m+1\}$ or $X = \{1, N\}$ we get a contribution of $-\epsilon/2$. In particular, $F(0)$ is not the zero vector in our Banach space.

We first show the fixed point equation has a solution using the norm

$$||e|| = \sum_{n=1}^{N} |e_n| + 2 \sum_{X:m} |e(X)|(n(X) - 2) \tag{36}$$

We will show that with the above norm, F is a contraction in a small ball about the origin. We have

$$\begin{aligned} ||F(e) - F(e')|| &\leq 2\epsilon \sum_{n} \sum_{j \in \partial(M-n)} |e(M\Delta(M-n)\Delta\{j, j+1\}) - e'(M\Delta(M-n)\Delta\{j, j+1\})| \\ &+ 2\epsilon \sum_{X:n(X)>2, X:m} \sum_{j \in \partial(X\Delta M)} |e(X\Delta\{j, j+1\}) - e'(X\Delta\{j, j+1\})| \\ &+ \sum_{X:n(X)>2, X:m} \sum_{s} |e_s\, e((X+s)\Delta(M+s)\Delta M) - e'_s\, e'((X+s)\Delta(M+s)\Delta M)| \end{aligned} \tag{37}$$

The terms in the first sum in the above are the terms one would get from the second sum for X with $n(X) = 2$. So together these two sums are

$$\begin{aligned} &= 2\epsilon \sum_{X:m} \sum_{j \in \partial(X\Delta M)} |e(X\Delta\{j, j+1\}) - e'(X\Delta\{j, j+1\})| \\ &+ 2\epsilon \sum_{X:m} \sum_{j \in \partial(X\Delta M)} |e(X) - e'(X)| = 2\epsilon \sum_{X:m} n(X)|e(X) - e'(X)| \end{aligned} \tag{38}$$

where we have used the change of variables $X \to X\Delta\{j, j+1\}$. If $n(X) = 2$, then $e(X) = e'(X)$. For the other X, $n(X) \geq 4$ and so $n(X) \leq 2(n(X) - 2)$. So the above is $\leq 4\epsilon||e - e'||$.

In the third sum in (37), we drop the constraint $n(X) > 2$ and do a change of variables $Y = (X+s)\Delta(M+s)\Delta M$. Then it is

$$\begin{aligned} &\leq \sum_{Y:m} \sum_{s} |e_s e(Y) - e'_s\, e'(Y)| \\ &\leq \sum_{Y:m} \sum_{s} |e_s|\, |e(Y) - e'(Y)| + \sum_{Y:m} \sum_{s} |e_s - e'_s|\, |e'(Y)| \\ &\leq \max\{||e||, ||e'||\}||e - e'|| \leq \delta||e - e'|| \end{aligned} \tag{39}$$

Thus F is a contraction on a ball of radius δ about the origin if δ and ϵ are small enough. Since $||F(0)||$ is of order ϵ, the above estimate also shows that F maps the ball $\{e : ||e|| < \delta\}$ back into itself if δ and ϵ are small enough.

As in the previous section, we can introduce a stronger norm. Let

$$||e|| = \sum_{n=1}^{N} |e_n|(K|\epsilon|)^{-w_n} + 2\sum_{X:m} |e(X)|(n(X)-2)(K|\epsilon|)^{-w(X)} \tag{40}$$

where $w(X)$ is defined as in the previous section and $w_n = w(M\Delta(M+n))$. We emphasize that this is a natural norm in the sense that the powers w_n and $w(X)$ are the lowest order in perturbation theory at which the corresponding terms get nonzero contributions. We leave it to the reader to check that the preceding estimates go through if $K|\epsilon| \leq 1$ and K is large enough. It is easy to see that $w_1 = m$ and the other w_n are even larger. Thus this stronger norm shows that the coefficients in the Fourier series of the dispersion relation are at least order ϵ^m.

We summarize our results on the droplet eigenstates in a theorem.

Theorem 1 *For the chain with periodic boundary conditions and N sites, we consider the sector with m down spins ($0 < m < N$). There is a constant $K > 0$ (independent of N) such that if $K|\epsilon| < 1$, then the fixed point equation has a solution in the above norm. The N eigenstates and eigenvalues determined by this solution through (22) and (26) have the lowest eigenvalues in this sector. (Except for $k = 0$ in which case it is the second lowest.) Letting $E_N(k)$ denote the eigenvalue for a chain with N sites corresponding to momentum k, there are constants d_s for $s \in \mathbf{Z}$ such that*

$$\lim_{N\to\infty} E_N(k) = 2 + \sum_{s=-\infty}^{\infty} d_s e^{iks} \tag{41}$$

The Fourier coefficients d_s are absolutely summable, and for $s \neq 0$ the coefficients are of order at most ϵ^m.

The existence of a solution to the fixed point equation implies that we have constructed N eigenvalues and eigenfunctions. To see that for each k the eigenvalue is the lowest eigenvalue in the sector of momentum k (or second lowest if $k = 0$) we argue as follows. We know the claim is true when $\epsilon = 0$. For a finite chain the eigenvalues are continuous in ϵ, and so our eigenvalue can cease being the lowest as ϵ is increased only by crossing another eigenvalue. At the value of ϵ where such a crossing would occur, our eigenvalue would be degenerate. One can then show that this implies the solution of the fixed point equation is not locally unique, contradicting the contraction mapping theorem. (More details may be found in [4].)

For a finite chain the eigenvalues $E_N(k)$ are only defined by a finite set of values which depends on N. But we can use use (26) to extend the definition to all k and so make sense of the limit in (41). The existence of the limit as $N \to \infty$ of the Fourier coefficients e_s follows by standard arguments. (See [4] for similar arguments.)

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